Hot Topics
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Recent results in connection to spherical spectral synthesis

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Spectral analysis and synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. A fundamental problem is to discover the structure of such spaces of functions, or more precisely, to find the appropriate class of basic functions, the so-called building blocks that serve as 'typical elements' of the space, i.e., a kind of basis. These turn out be the so-called *exponential monomials*. Consider the space of all complex-valued continuous functions on the real line with respect to the pointwise linear operations and to the topology of uniform convergence on compact sets. Suppose that a closed translation invariant subspace of this space is given. This subspace may or may not contain an exponential monomial. If it does, then we say that *spectral analysis* holds for this subspace. In such a situation, the exponential function itself is contained in this subspace. The complex number, characterizing this exponential function, can be considered as a *spectral value*. The complete description of the subspace, however, means that all the exponential monomials corresponding to the spectral exponentials and their multiplicities characterize the subspace: their linear hull is dense in this subspace. If so, we say that *spectral synthesis* holds for this subspace.

L. Schwartz proved that any closed translation invariant linear space of continuous functions on the reals is synthesizable from its exponential monomials. The above construction can obviously be generalized: instead of the topological group of the reals, the set of all continuous, complex-valued functions on a locally compact Abelian group (equipped with the pointwise linear operation and with the topology of uniform convergence on compact sets) can be considered. Furthermore, exponential functions and exponential polynomials should also be defined in this setting. After this, the problems of spectral analysis and synthesis can be formulated: is it true that any nonzero closed translation invariant subspace of the above space contains an exponential (spectral analysis), and is it true that in any subspace of this kind, the linear hull of all exponential monomials is dense (spectral synthesis)?

Later, L. Schwartz asked whether the investigated spaces have this property when $n > 1$. D. Gurevič provided counterexamples of spectral analysis and synthesis for the spaces $\mathscr{E}^n$ (all infinitely many differentiable functions on $\mathbb{R}^n$), $\mathscr{H}$ (all entire functions over $\mathbb{C}^n$) and $\mathscr{D}'$ (distributions over $\mathbb{R}^n$). In particular, he also constructed a homogeneous system of convolution type equations such that exponential polynomial solutions of this system are not dense in the space of all their solutions.

To prove affirmative results for higher dimensional cases, in L. Székelyhidi replaced 'translation invariance' by 'invariance with respect to some compact group of automorphisms'. For this purpose, the basic concepts and notions of spectral analysis and synthesis need to be introduced with the help of Gelfand pairs. The role of exponentials then is played by spherical functions. After elaborating the general theory, in the special case where the basic group is $\mathbb{R}^n$ and the compact group of automorphisms is $\mathrm{SO}(n)$.

Following L. Székelyhidi, with the aid of Gelfand pairs and K-spherical functions, K-synthesizability of K-varieties can be described. In this talk we aim to contribute to this direction in the special case when K is the symmetric group of order d.